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On the Periodic Intermediate Long Wave Equation

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by

M. J. Ablowitz, A. S. Fokas, J. Satsuma*
Clarkson College of Technology
Department of Mathematics and Computer Science
Potsdam, New York 13676

and

Harvey Segur
Aeronautical Research Associates of Princeton, Inc.
50 Washington Road, P.O. Box 2229
Princeton, New Jersey 08540

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*Present address: General Education, Miyazaki Medical College, Kiyotaki, Miyazaki, 889-16, Japan.

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ABSTRACT

We discuss some properties of a certain physically interesting nonlinear integro-differential equation with periodic boundary conditions. It is the natural periodic analogue of the intermediate long wave equation, and it provides a periodic analogue of the Benjamin-Ono equation in the appropriate limit. Due to the speciality integral operator, the equation admits a Bäcklund transformation, an infinity of motion constants, etc. Two simple periodic solutions are exhibited. Finally we note that the equation may be transformed into more than one kind of bilinear equation.

The so-called intermediate long wave (I.L.W.) equation (Joseph, 1977; Kubota, Ko and Dobbs, 1978) can be written in the form

$$u_t + \frac{1}{\delta} u_x + 2uu_x + (T\,u)_{xx} = 0 \tag{1}$$

on $-\infty < x < \infty$, where

$$T\,u = -\frac{1}{2\delta} \int_{-\infty}^{\infty} \coth \frac{\pi}{2\delta}(x-y)\, u(y)\, dy \tag{2}$$

and the integral is evaluated in the principal-value sense. The equation can be solved on $(-\infty,\infty)$ via an inverse scattering transform (IST) (Kodama, Satsuma and Ablowitz, 1981), and has the analytical structure associated with such equations (Joseph and Egri, 1978; Satsuma, Ablowitz and Kodama, 1979).

The physical derivation of (1), (2) as a model of the evolution of long internal waves of moderate amplitude assumes that $u(x)$ has a classical Fourier transform, and that u vanishes as $|x| \to \infty$. Even so, one may ask whether (1), (2) admit spatially periodic solutions. This was done by Joseph and Egri (1978), Chen and Lee (1979), and Nakamura and Matsuno (1980), using formal algebraic methods. Unfortunately, the solutions so obtained either contain errors or are subject to a limitation that was obscured by these formal methods.

Alternatively, one may seek an evolution equation for long internal waves of moderate amplitude that are spatially periodic. Then a derivation similar to the usual one leads to (1), but in the periodic case (2) is replaced with

$$Tu = \frac{1}{2L} \int_{-L}^{L} \bar{T}(x-\varsigma;\delta,L)\, u(\varsigma)\, d\varsigma \tag{3a}$$

where

$$\bar{T}(x;\delta,L) = -\frac{2K}{\pi}\left[Z\left(\frac{Kx}{L}\right) + dn\left(\frac{Kx}{L}\right) cs\left(\frac{Kx}{L}\right)\right]. \tag{3b}$$

[Actually, the physical derivation naturally leads to the Fourier representation of $\bar{T}$, given in (3c), which is then transformed into (3b).] This derivation also requires that $\int_{-L}^{L} u\, dx = 0$, which we may impose on (1) with (3) without loss of generality. In (3b), K denotes the complete elliptic integral of the first kind, $Z(a)$ is Jacobi's Zeta function, and $dn(a)$, $cs(a)$ are Jacobian elliptic functions. These all have modulus k, determined by the condition that $K'(k)/K(k) = \delta/L$, where $K'(k)$ is the associated elliptic integral of the first kind. (All of these functions are discussed by Byrd and Friedman, 1971.) The purpose of this note is to discuss some of the mathematical structure of (1) with (3).

An alternative, but very useful, representation of $\tilde{T}$ is its Fourier series,

$$\tilde{T}(x;\,\delta,L) \;=\; i \sum_{n\neq 0} \coth\left(\frac{n\pi\delta}{L}\right) \exp\left(\frac{in\pi x}{L}\right), \tag{3c}$$

so that

$$Tu \;=\; i \sum_{n\neq 0} \coth\left(\frac{n\pi\delta}{L}\right) \hat{u}_n \exp\left(\frac{in\pi x}{L}\right), \tag{3d}$$

where $\{\hat{u}_n\}$ are the Fourier coefficients of u. This representation follows from the identities (c.f., Byrd and Friedman, 1971, #905.01 and 908.51)

$$Z(a) \;=\; \frac{2\pi}{K} \sum_{m=1}^{\infty} \frac{q^m}{1-q^{2m}} \sin\left(\frac{m\pi a}{K}\right)$$

$$\mathrm{cs}(a)\,\mathrm{dn}(a) \;=\; \frac{\pi}{2K} \cot\frac{\pi a}{2K} - \frac{2\pi}{K} \sum_{m=1}^{\infty} \frac{q^m}{1+q^m} \sin\left(\frac{m\pi a}{K}\right)$$

where $q = \exp(-\pi K'/K) = \exp(-\pi\delta/L)$, and from the formal representation (Gel'fand and Shilov, 1964, p. 32)

$$\cot \frac{\theta}{2} = 2 \sum_{m=1}^{\infty} \sin m\theta \; .$$

The usual operator on $(-\infty, \infty)$ may be recovered simply by replacing the sum in (3d) with an integral and rescaling. Similarly, we may recover (2) from (3) by letting $L \to \infty$, δ fixed. Then $k \to 1$, $K \to \pi L/2\delta$, and one may show that

$$\frac{1}{2L} \tilde{T}(x;\ \delta,\ L{\to}\infty) \quad \to \quad \frac{1}{2\delta} \coth \frac{\pi x}{2\delta} \; , \tag{4a}$$

which reproduces (2). On the other hand, for $\delta \to \infty$, L fixed, we have $k \to 0$, $K \to \pi/2$, and

$$\tilde{T}(x;\ \delta{\to}\infty,\ L) \quad \to \quad -\cot \frac{\pi x}{2L} \; . \tag{4b}$$

This is the well-known Hilbert kernel on $(-L,L)$. With this kernel, (1) is the natural periodic extension of the Benjamin-Ono equation (Benjamin, 1967; Ono, 1975). As one would expect, (1) with (3) reduces to the (periodic) KdV equation if $\delta \to 0$, L fixed.

If $u(x)$ is periodic with period 2L and with zero mean, then (Tu) according to (2) and to (3) are identical. That this is so may be seen by rewriting (2) in its Fourier transform representation, and recalling that the F.T. of a periodic function is a sum of Dirac delta functions (Gel'fand and Shilov, 1964). Thus (1) with (3) may be

regarded simply as the most natural way to write the I.L.W. equation when periodic solutions are of interest.

The operator T given in (3) is the most general periodic operator we have found which satisfies the "T-conditions":

$$T(u\,Tv + v\,Tu) = Tu\,Tv - uv \tag{T1}$$

$$\int_{-\infty}^{\infty} (u\,Tv + v\,Tu)\,dx = 0\ , \tag{T2}$$

where u,v have zero mean. We find that (T1), (T2) are necessary conditions on the T operator in order for (1) to have more than the standard number of conserved quantities. With these conditions the above evolution equations can be expected to be in the IST class. Condition (T2) follows from the fact that T(x) is an odd function. To establish (T1) we use the representation (3c). Calling $\hat{T}_n$ = icoth $n\pi\delta/L$, and $\hat{u}_n$, $\hat{v}_n$, the Fourier coefficients of u, v respectively, assuming $\hat{u}_0 = \hat{v}_0 = 0$ (i.e., zero mean) and using the convolution theorem, then

$$T(u\,Tv) = \sum_{n=-\infty}^{\infty} \hat{f}_n \sum_{m=-\infty}^{\infty} \hat{f}_m \, \hat{v}_{n-m} \, \hat{u}_{n-m} \exp\left(\frac{in\pi x}{L}\right)$$

$$= \sum_{m} \hat{u}_m \sum_{n} \hat{f}_n \, \hat{f}_{n-m} \, \hat{v}_{n-m} \exp\left(\frac{in\pi x}{L}\right)$$

$$= -\sum_{m} \hat{u}_m \sum_{n} \hat{v}_{n-m} \exp\left(\frac{in\pi x}{L}\right)$$

$$-\sum_{m} \hat{u}_m \sum_{n} \hat{f}_m \, \hat{f}_n \, \hat{v}_{n-m} \exp\left(\frac{in\pi x}{L}\right)$$

$$+\sum_{m} \hat{u}_m \sum_{n} \hat{f}_m \, \hat{f}_{n-m} \, \hat{v}_{n-m} \exp\left(\frac{in\pi x}{L}\right)$$

$$= -u(x)\, v(x) - T(v\,Tu) + (Tu)(Tv) \qquad (5)$$

where we have used the identity

$\coth A \coth B = 1 + \coth(A-B)(-\coth A + \coth B)$.

The order of summation in (5) may be interchanged if $\sum|\hat{u}_n|$, $\sum|\hat{v}_n|$ exist.

The need for the T-conditions can be seen from the following. The usual I.L.W. equation on$(-\infty,\infty)$ was considered by Satsuma, Ablowitz, and Kodama (1979). They showed that the derivations of the constants of motion and of the Bäcklund transformation do not depend on the specific kernel of T, so long as T satisfies conditions (T1) and (T2). This implies that these formulae will remain valid for equation (1) with (3) since the T conditions are satisfied. Actually one easily verifies (by differentiation) that the constants of motion given by Satsuma, Ablowitz and Kodama (1979) are also constants of motion of (1) with (3).

One may use the generalized Miura transformation, $u = (\hat{\lambda}-v_x+iTv_x+\hat{\mu}e^{iv})/2$ to derive a generalization of the so-called modified I.L.W. equation. Namely if $\tilde{K}[u] = 0$ represents (1), and

$\tilde{M}[v] = v_t + (1/\delta+\hat{\lambda})v_x + (Tv)_{xx} + \hat{\mu}v_x e^{iv} + iv_x(Tv)_x = 0$,

then using the T-conditions we have:

$$\tilde{K}[u] = \mathcal{O}\tilde{M}[v] , \tag{6}$$

where

$$\mathcal{O} = (-\partial_x + iT\partial_x + i\hat{\mu}\, e^{iv}) .$$

Similarly it can be proven, by using the results of Fokas and Fuchssteiner (1980) (which also apply to the above Miura-type transformation) that (1) admits an auto-Bäcklund transformation if and

only if (T1), (T2) are satisfied.

The IST pair used by Kodama, Satsuma and Ablowitz (1981) is also valid here. However, the simple compatibility argument does not indicate the importance of the T-conditions. Namely, consider a linear scattering problem and a sequence of associated time evolutions of the form:

$$i\,\psi_x^+ + (u-\lambda)\,\psi^+ = \mu\,\psi^- , \tag{7a}$$

$$\psi_t^\pm = Q_n^\pm\,\psi^\pm , \qquad n = 1, 2, \ldots \tag{7b}$$

(7a) is to be thought of as a differential Riemann-Hilbert problem to find $\psi^\pm$ (i.e., $\psi^\pm$ are the boundary values of certain analytic *functions) given an appropriate funtion* $u(x)$. For each n, compatibility of (7a,b) yields an evolution equation; namely, requiring $\psi_{xt}^\pm = \psi_{tx}^\pm$, and setting all coefficients of ψ^+, ψ^-, ψ_x^-, ψ_{xx}^-, etc., to zero after using (7a) to eliminate derivatives of ψ^+, gives an algorithmic procedure to determine compatible equations. The equivalent to the first two equations of KdV hierarchy are obtained as follows. First, $Q_1^\pm = \partial_x + A^\pm$, whereupon we find $A^+ - A^- = 0$. Taking $A^\pm = A_0 = \text{const}$, the compatible evolution equation is $u_t = u_x$. Second, $Q_2^\pm = i\partial_x^2 + iB^\pm\partial_x + iA^\pm$, we find $B^+ - B^- = 0$, $A^+ - A^- = -2iu_x$. Taking $B^\pm = B_0 = i(2\lambda+1/\delta) = \text{const}$, $A^\pm = \pm iu_x - (Tu)_x$, the compatible evolution equation is (1), without need for the T-conditions. The underlying reason why such conditions

must be added, and whether, in fact, (3) is the most general singular integral operator satisfying (T1), (T2) are open questions. Matsuno (1980) has given a different algorithm to derive such a hierarchy of equations.

Next we consider some special solutions of (1) with (3). For a wave of permanent form, $\partial_t \to -c\partial_x$, and (1) may be integrated once to

$$\left(\frac{1}{\delta} - c\right) u + u^2 + (Tu)_x + A = 0 \tag{8}$$

with A constant. It has been claimed (Joseph and Egri, 1978; Chen and Lee, 1979) that

$$u(x - ct) = \frac{-(n\pi/L)\sinh(n\pi\delta/L)}{\cosh(n\pi\delta/L) + \cos[n\pi(x - ct)/L]} \tag{9a}$$

is a solution of (8). Its Fourier series representation is

$$u(\zeta) = -\frac{n\pi}{L}\left[1 + 2\sum_{1}^{\infty} (-1)^m \exp\left(-\frac{mn\pi\delta}{L}\right) \cos\left(\frac{mn\pi\zeta}{L}\right)\right] . \tag{9b}$$

That (9) does not solve (8) may be seen by computing $\partial u/\partial\delta$ from (9a) and (9b), and Tu from (3d), because it turns out that $\partial u/\partial\delta = -\partial(Tu)/\partial x$. The correct solution is

$$u(x - ct) = \frac{-(n\pi/L)\ \sinh(n\pi\delta/L)}{\cosh(n\pi\delta/L) + \cos[(n\pi/L)(x - ct - i\gamma)]} \tag{10a}$$

where $\gamma < -\delta$. Its Fourier series is

$$u(\zeta) = \frac{2n\pi}{L} \sum_{1}^{\infty} (-1)^{m} \sinh\left(\frac{mn\pi\delta}{L}\right) \exp\left(\frac{imn\pi(\zeta - i\gamma)}{L}\right), \tag{10b}$$

and its validity may be verified by computing $\partial u/\partial\delta$ from (10a) and (10b). This solution is complex-valued.

Periodic solutions may be obtained systematically via Hirota's method, revising slightly the methods of Chen and Lee (1979), Nakamura and Matsuno (1980), and Satsuma and Ablowitz (1980). Let $f^{\pm}(x) \equiv f(x \mp i\delta)$, let $f(x)$ be periodic ($2L$) and let $f(z)$ be analytic in the rectangle: $-L \leq \mathrm{Re}(z) \leq L$, $-\delta \leq \mathrm{Im}(z) \leq \delta$. Then by integrating $\int \tilde{T}(x-\zeta+i\delta)\ f(\zeta)d\zeta$ around this rectangle, one finds that

$$\frac{1}{2L} \int_{-L}^{L} \tilde{T}(x - z; \delta, L)[f^{+}(z) - f^{-}(z)]\,dz = i[f^{+}(x) + f^{-}(x)] + J_0\,, \tag{11}$$

where J_0 is an unimportant constant. It follows that if

$$u(x) = i[f^{+}(x) - f^{-}(x)] \tag{12}$$

where

$$f^{\pm}(x) = \partial_x (\log F^{\pm}) ,$$

and if f(x) is properly analytic, then (1) with (3) becomes a bilinear equation,

$$\left(i D_t + \frac{i}{\delta} D_x - D_x^2 + A\right) F^+ \cdot F^- = 0 \qquad (13)$$

with the usual notation [e.g., $D_x\ a.b \equiv (\partial_x - \partial_{x'})\ a(x)b(x')|_{x'=x}$], and $A = A(t)$. We emphasize that only those solutions of (13) that are analytic in the rectangle yield periodic solutions of (1), a fact that was overlooked previously.

The simplest real-valued solution of (1) with (3) was given by Nakamura and Matsuno (1980). It may be written in the form

$$u(\zeta; m) = \frac{iK(m)}{L}\left[Z\left(\frac{K(m)}{L}(\zeta - i\delta); m\right) - Z\left(\frac{K(m)}{L}(\zeta + i\delta); m\right)\right] \qquad (14a)$$

where $\zeta = x - ct + x_0$, and m is the modulus. The analyticity condition is that

$$\frac{\delta}{L} < \frac{K'(m)}{K(m)} , \qquad (14b)$$

or, because $\delta/L = K'(k)/K(k)$, that the modulus of T exceed the modulus of the solution. There seems to be no simple formula for the speed of the wave.

Finally, we note that (13) is not the only "bilinear" equation that may be obtained from (1), (3). For example, let $f(z)$ have a real period ($2L$), and be analytic in $-L \leq Re(z) \leq L$, $-\delta \leq Im(z) \leq \delta$ except for two poles at $z = z_0$ and z_0^* $[0 < Im(z_0) < \delta]$ with residues b and b^*. Then the integral of $\int \tilde{T}(x-\zeta+i\delta)\, f(\zeta)d\zeta$ around the rectangle yields

$$\frac{1}{2L}\int_{-L}^{L} \tilde{T}(x-z)[f^+(z) - f^-(z)]\,dz = i[f^+(x) + f^-(x)] + J_0 - \frac{\pi i}{L}\left[b\,\tilde{T}(x-z_0+i\delta) + b^*\,\tilde{T}(x-z_0^*+i\delta)\right] \qquad (15)$$

instead of (11). In this case, (12) changes (1) with (3) into

$$\left\{i D_t + \frac{i}{\delta} D_x - D_x^2 + A + \frac{\pi}{L}\left[b\,\partial_x \tilde{T}(x-z_0+i\delta) + b^*\,\partial_x \tilde{T}(x-z_0+i\delta)\right]\right\} F^+ \cdot F^- = 0 . \qquad (16)$$

The analyticity requirement is that $F(z)$ should be analytic in the usual rectangle except for simple branch points at z_0 and z_0^*, so that $f^{\pm}$ has poles. We have not determined whether (15) yields any solutions of (1),(3) that are not available via (13).

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